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Nonlinear theory of geostrophic adjustment. Part 2. Two-layer and continuously stratified primitive equations

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This paper continues the work started in Part 1 (Reznik, Zeitlin & Ben Jelloul 2001) and generalizes it to the case of a stratified environment. Geostrophic adjustment of localized disturbances is considered in the context of the two-layer shallow-water and continuously stratified primitive equations in the vertically bounded and horizontally infinite domain on the f-plane. Using multiple-time-scale perturbation expansions in Rossby number Ro we show that stratification does not substantially change the adjustment scenario established in Part 1 and any disturbance of well-defined scale is split in a unique way into slow and fast components with characteristic time scales f_0^{-1} and $(f_0 Ro)^{-1}$ respectively, where f_0 is the Coriolis parameter. As in Part 1 we distinguish two basic dynamical regimes: quasi-geostrophic (QG) and frontal geostrophic (FG) with small and large deviations of the isopycnal surfaces, respectively. We show that the dynamics of the FG regime in the two-layer model depends strongly on the ratio of the layer depths. The difference between QG and FG scenarios of adjustment is demonstrated. In the QG case the fast component of the flow essentially does not 'feel' the slow one and is rapidly dispersed leaving the slow component to evolve according to the standard QG equation (corrections to this equation are found for times $t \gg (f_0 Ro)^{-1}$). In the FG case the fast component is a packet of inertial oscillations produced by the initial perturbation. The space-time evolution of the envelope of inertial oscillations obeys a Schrödinger-type modulation equation with coefficients depending on the slow component. In both QG and FG cases we show by direct computations that the fast component does not produce any drag terms in the equations for the slow component; the slow component remains close to the geostrophic balance. However, in the continuously stratified FG regime, as well as in the two-layer regime with the layers of comparable thickness, the splitting is incomplete in the sense that the slow vortical component and the inertial oscillations envelope evolve on the same time scale.

1. Introduction

In the first part of this work (Reznik, Zeitlin & Ben Jelloul 2001, hereafter referred to as Part 1) we developed a theory of nonlinear geostrophic adjustment of arbitrary localized finite-energy disturbances in the framework of the non-dissipative rotating shallow-water (RSW) dynamics. The only assumptions made were the well-defined

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scale of the disturbance and the smallness of the Rossby number Ro. The latter assumption allows to use multi-time-scale expansions for solving the initial-value problem. It was shown that velocity and pressure fields are split in a unique way into slow and fast components with characteristic time scales f_0^{-1} and $(f_0 Ro)^{-1}$ respectively, where f_0 is the Coriolis parameter. The slow component is not influenced by the fast one and remains close to the geostrophic balance. The algorithm of initialization of both components follows by construction. The scenario of adjustment depends on the characteristic scale and/or initial relative elevation of the free surface. For small relative elevations the evolution of the slow motion is governed by the well-known quasi-geostrophic (QG) dynamics for times $t \leqslant (f_0 Ro)^{-1}$ and modifications to this dynamics for longer times $t \leqslant (f_0 Ro^2)^{-1}$ were found. The fast component consists mainly of linear inertia–gravity waves rapidly propagating outward from the initial disturbance. For large relative elevations the slow vortex field is governed by the frontal geostrophic (FG) dynamics equation. In this case the fast component is a spatially localized packet of inertial oscillations evolving on the background of the slow component of the flow. The envelope of the packet obeys a Schrödinger-type equation with coefficients depending on the even slower vortex motion.

While the QG results corroborated the 'standard wisdom' view of adjustment, the FG results showed a possibility of incomplete or delayed adjustment due to inertial oscillations coexisting with the slow component of motion.

The question we address in the present paper is how stratification modifies these results. The simplest way to introduce stratification effects is to consider layered models. We, therefore, start our analysis with the standard rotating two-layer shallow-water model (which will be abbreviated as 2RSW in what follows) with rigid lid and flat bottom boundary conditions. We thus have the baroclinic interface displacement instead of the barotropic free-surface elevation.

As in the RSW case a slow-dynamics reduction of the model by implicit filtering of the internal inertia–gravity waves (IGW) via exclusive use of the slow ('vortical') time scale $(f_0 Ro)^{-1}$ is standard and widely applied (cf. e.g. Pedlosky 1982; Gent & McWilliams 1983*a*, *b*). Similarly to the RSW case, in the two-layer (or, more generally, multi-layer) models the geostrophic balance condition allows different dynamical regimes depending on the value of the Burger number and the ratio of the layer depths. The standard QG regime corresponds to small deviations of the isopycnals from their equilibrium positions and to typical horizontal scales of the order of the Rossby deformation radius R_R. The frontal geostrophic (FG) regime corresponds to large deviations of the isopycnal surfaces under the condition that the typical horizontal scale of the flow greatly exceeds the deformation radius. The FG regime in the two-layer model was first introduced, again by using exclusively the slow time scale, by Cushman-Roisin, Sutyrin & Tang (1992). Analogous regimes in the two-layer ocean with a sloping bottom were analysed by Swaters (1993). In both works the upper layer was assumed to be much thinner than the lower one; the corresponding model will be referred to as inhomogeneous (FGI) in what follows. Later Benilov & Reznik (1996) classified all possible strongly nonlinear regimes in a two-layer ocean of constant depth (some of these regimes were studied independently by Stegner & Zeitlin (1996) in the context of near-axisymmetric solitary vortices). A frontal regime with the depths of the layers of the same order, which we call homogeneous (FGH) below was found. A complete classification of the two-layer frontal regimes including the effects of planetary sphericity and variable bottom topography was given by Karsten & Swaters (1999). The same authors (Karsten & Swaters 2000*a*, *b*) explored the stability of various two-layer FG sub-regimes on the beta-plane.

In what follows we propose a full perturbative derivation of the slow dynamics equations for the two-layer QG and FG cases, instead of the 'filtered' derivation, by imposing an *ad hoc* time scale. We consider arbitrary non-balanced initial disturbances of well-defined horizontal and vertical scales under the single condition that the Rossby number ($Ro = \epsilon$, in what follows) is small, and analyse the dynamics using multiple-time-scale asymptotic expansions in ϵ. We show that, as in the RSW case, the above-mentioned QG and FG slow-dynamics equations follow from the removal of resonances in the fast dynamics and, once the resonances are eliminated, the fast component can be completely quantified. We calculate explicitly the Reynolds stresses due to the fast component and demonstrate that they vanish at the first three orders of the perturbation theory. Corrections to the standard QG dynamics for times much longer than $(f_0 Ro)^{-1}$ are obtained. Our construction provides an algorithm for initialization of both fast and slow variables. In the FG regime, it turns out that the fast component accompanies the slow one in the form of inertial oscillations. Some recent experimental results may be explained by this incomplete adjustment in the two-layer systems (Stegner, Bouruet-Aubertot & Pichon 2003).

We then consider the continuously stratified case in the same geometry (the N-layer generalizations are straightforward). For simplicity we use the hydrostatic primitive equations (HSPE); therefore vertically propagating inertia waves are excluded and the model is of the shallow-water type. Here we again consider the standard QG regime (cf. Pedlosky 1982) and the FG regime, whose slow-dynamics version for continuous stratification was introduced by Benilov (1993) and turns out to be a generalization of the FGH regime in 2RSW. We do not include the analysis of regimes intermediate between QG and FG (Romanova & Zeitlin 1984; Stegner & Zeitlin 1999) in the present paper; it may be done along the same lines as in Part 1 for RSW.

The method of multiple-time scale expansions in Rossby number allows us to prove asymptotic validity of the above-mentioned balanced models, to find corrections to these models for longer times and to specify the adjustment scenarios. A key element of our approach is the initial-value problem setting and the radiation boundary conditions for waves which means that most of the resonances appearing in the (triple-) periodic box geometry are avoided. The absence of the fast-motion drag in the slow equations is proved and the modulation equation for the inertial oscillations in the FG regime is obtained.

It should be noted that, as in Part 1, below we limit considerations to vortex-like initial perturbations having a single horizontal scale. The calculations here are closely related to those in Part 1, especially regarding the QG regime. We, thus limit ourselves to a presentation of the main results and a brief description of the method in the QG regime. Full proofs and detailed calculations are available as a supplement to the online version, or from the authors or the Journal of Fluid Mechanics Editorial Office, Cambridge. The FG calculations are given in more detail. We present the main results first so readers not interested in subsequent proofs may skip the details. We do not repeat here the classical references on geostrophic adjustment which may be found in Part 1. The same notation as in Part 1 is kept whenever possible.

The paper is organized as follows. In § 2 we present an analysis of the 2RSW model, in the QG (§ 2.2) and the FG (§ 2.3) regimes. Section 3 contains the analysis of the continuously stratified case in the framework of the HSPE with QG being treated in § 3.2 and FG in § 3.3. Finally, a discussion and comparison with the existing literature are presented in § 4.

2. Geostrophic adjustment in the two-layer shallow water model on the f-plane

2.1. *Preliminaries*

We consider a non-dissipative fluid on the f-plane contained between a rigid lid (at $z=0$) and a rigid bottom (at $z=-H$; topographic effects may be easily introduced). The unperturbed depths of the upper and lower layers are H_1 and H_2, respectively; $H_1+H_2=H$.

The equations of motion for a two-layer rotating shallow-water system (2RSW) are the horizontal momentum equations:

$$\partial_t \boldsymbol{v}_i + \boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_i + \frac{1}{\rho_i}\nabla \pi_i = 0, \quad i = 1, 2 \tag{2.1}$$

(no summation over i; the combination $i+1$ is understood modulo 2 everywhere below); and the mass conservation equations in each layer:

$$\partial_t(H_i - (-1)^{i+1}\eta) + \nabla \cdot (\boldsymbol{v}_i(H_i - (-1)^{i+1}\eta)) = 0, \quad i = 1, 2, \tag{2.2}$$

where f is the Coriolis parameter which is equal to f_0 in the f-plane approximation (to be adopted below unless otherwise stated), $\boldsymbol{v}_i = (u_i(x, y, t), v_i(x, y, t))$ is the two-dimensional velocity field in each of the two layers, ρ_i is the density of the layers, η is the vertical displacement of the interface, π_i are defined with the help of the full pressure fields P_i in each layer:

$$P_i = -\rho_i g z + (i-1)(\rho_1 - \rho_2) g H_1 + \pi_i, \tag{2.3}$$

g is the acceleration due to gravity (g becomes the reduced gravity g' below). Here and below $\partial^n_{abc\ldots}$ denotes the nth partial derivative with respect to $a, b, c, \ldots$, $\nabla = (\partial_x, \partial_y)$ in this section and $\hat{\boldsymbol{z}}$ is the vertical unit vector.

The potential vorticity (PV) conservation equations in each layer readily follow:

$$(\partial_t + \boldsymbol{v}_i \cdot \nabla)\Pi_i = 0, \quad \Pi_i = \frac{\zeta_i + f}{H_i - (-1)^{i+1}\eta}, \tag{2.4}$$

where $\zeta_i = \hat{\boldsymbol{z}} \cdot \nabla \wedge \boldsymbol{v}_i$ is the relative vorticity in each layer. From the dynamical boundary condition on the interface

$$P_1 = P_2|_{z=-H_1+\eta} \tag{2.5}$$

it follows that

$$(\rho_2 - \rho_1) g \eta = \pi_2 - \pi_1. \tag{2.6}$$

The vertical velocity should vanish at the top and bottom.

The following parameters:

$$N = 2\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \quad d = \frac{H_1}{H_2}, \quad g' = gN, \quad \bar{H} = \frac{H_1 H_2}{H_1 + H_2}, \quad \bar{\rho} = \frac{1}{2}(\rho_1 + \rho_2)$$

will be used below for compactness.

2.2. *The QG regime*

2.2.1. *Definitions*

Assuming in this Subsection that $d = O(1)$ we introduce the following QG-scaling: horizontal velocity scale U, horizontal spatial scale $L \sim R_R = \sqrt{g'\bar{H}}/f_0$, where R_R is the baroclinic Rossby deformation radius, pressure scale $P = \bar{\rho} f_0 U L$ and the scale of the interface variations $\eta^* = \epsilon \bar{H}$. The time scale is f_0^{-1}. Introducing the (order one) parameters $\bar{h}_i = H_i/(H_1 + H_2)$, $i = 1, 2$ we rewrite the horizontal momentum and mass conservation equations in the following non-dimensional form:

$$\partial_t \boldsymbol{v}_i + \epsilon \boldsymbol{v}_i \cdot \nabla \boldsymbol{v}_i + \hat{\boldsymbol{z}} \wedge \boldsymbol{v}_i + \nabla \pi_i = 0, \quad i = 1, 2; \tag{2.7}$$

$$\partial_t (1 - (-1)^{i+1} \epsilon \bar{h}_{i+1} \eta) + \nabla \cdot ((1 - (-1)^{i+1} \epsilon \bar{h}_{i+1} \eta) \boldsymbol{v}_i) = 0, \quad i = 1, 2. \tag{2.8}$$

Here, in order to simplify the formulae we adopt the oceanographic context and suppose that the densities of the layers are close to each other (and to the mean density). The corresponding density ratios may be easily restored in front of the pressure gradient terms here and below. The non-dimensional PVs are

$$\Pi_i = \frac{\epsilon \zeta_i + 1}{1 - (-1)^{i+1} \bar{h}_{i+1} \epsilon \eta}, \tag{2.9}$$

where ζ_i here and below we denote the relative vorticities by $\zeta_i = \partial_x v_i - \partial_y u_i$; $\boldsymbol{v}_i = (u_i, v_i)$. The non-dimensional version of (2.6) is

$$\pi_2 - \pi_1 = \eta. \tag{2.10}$$

The perturbative expansions for the velocity and interface displacement fields are

$$\left.\begin{aligned} \boldsymbol{v}_i &= \boldsymbol{v}_i^{(0)}(x, y; t, t_1, t_2, \ldots) + \epsilon \boldsymbol{v}_i^{(1)}(x, y; t, t_1, t_2, \ldots) + \ldots, \\ \eta &= \eta^{(0)}(x, y; t, t_1, t_2, \ldots) + \epsilon \eta^{(1)}(x, y; t, t_1, t_2, \ldots) + \ldots, \end{aligned}\right\} \tag{2.11}$$

where $t_1, t_2 \ldots$ scale as $(\epsilon f_0)^{-1}, \epsilon^{-2} f_0^{-1}, \ldots$, and each dynamical variable at each order may be uniquely split into the slow (denoted below by an overbar) and fast (denoted by a tilde) part defined, correspondingly, as the average over the fast time t and the fluctuation around it. In what follows we are looking for a perturbative solution of the Cauchy problem with initial conditions

$$\boldsymbol{v}_i(t = 0) = \boldsymbol{v}_{I_i}, \quad \eta(t = 0) = \eta_I \tag{2.12}$$

for the system (2.1), (2.2) under the QG scaling.

2.2.2. *The main results*

Each field is represented as a sum of slow and fast components. For example, the velocity field is

$$\boldsymbol{v}_i = \sum_{n=0}^{\infty} \epsilon^n \bar{\boldsymbol{v}}_i^{(n)}(x, y; t_1, t_2, \ldots) + \sum_{n=0}^{\infty} \epsilon^n \tilde{\boldsymbol{v}}_i^{(n)}(x, y; t, t_1, t_2, \ldots), \tag{2.13}$$

and correspondingly for π_i, η. The representation (2.13) is unique since the fast components $\tilde{\boldsymbol{v}}_i^{(n)}$ are defined to have zero mean over the fast time t:

$$\left\langle \tilde{\boldsymbol{v}}_i^{(n)} \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \, \tilde{\boldsymbol{v}}_i^{(n)} = 0. \tag{2.14}$$

We are able to demonstrate (up to the third order in ϵ) that the slow and the fast components are dynamically split and obey their own evolution equations with uniquely defined initial conditions. The initialization procedure for each component is given order by order in ϵ. The slow component is described (up to order-ϵ terms) by the pair of coupled equations for the pressure and the interface displacement variables $\bar{\pi}_i = \bar{\pi}_i^{(0)} + \epsilon \bar{\pi}_i^{(1)}$, $\bar{\eta} = \bar{\eta}^{(0)} + \epsilon \bar{\eta}^{(1)}$:

$$\begin{aligned} \frac{\mathrm{D}_i}{\mathrm{D}t_1} [\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta} + \epsilon (-1)^{i+1} \bar{h}_{i+1} \bar{\eta} (\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta}) \\ - \epsilon \nabla \bar{\pi}_i \cdot \nabla (\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta}) - 2\epsilon J(\partial_x \bar{\pi}_i, \partial_y \bar{\pi}_i)] = 0, \end{aligned} \tag{2.15}$$

where

$$\frac{\mathrm{D}_i}{\mathrm{D}t_1}(\ldots) := \partial_{t_1}(\ldots) + J\left(\bar{\pi}_i - \epsilon\frac{(\nabla\bar{\pi}_i)^2}{2}, \ldots\right), \quad i = 1, 2 \tag{2.16}$$

and $\bar{\pi}_2 - \bar{\pi}_1 = \bar{\eta}$.

The fast component is the inertia–gravity wave packet formed by the initial conditions and described by the wave equation for $\tilde{\eta} = \tilde{\eta}^{(0)} + \epsilon\tilde{\eta}^{(1)}$:

$$-\frac{\partial^2\tilde{\eta}}{\partial t^2} - \tilde{\eta} + \nabla^2\tilde{\eta} = \epsilon\mathscr{R}(x, y; t, t_1, \ldots) \tag{2.17}$$

with the known right-hand side produced by the nonlinear interaction of the lowest-order fast field with itself and with the slow component. The key point is that $\mathscr{R}$ does not contain resonant terms and has zero mean in the sense of (2.14) and, therefore, the field $\tilde{\eta}$ and all other fast fields consist of IGW propagating outward of the localized initial perturbation and decay in time at any given spatial location. This is why the fast-component drag upon the slow fields is absent.

When $\epsilon = 0$ equations (2.15), (2.16) are reduced to the standard QG system for the 2-layer model describing the QG motion on times of the order of $(\epsilon f_0)^{-1}$. Equation (2.15) is applicable on much longer times of the order $(\epsilon f_0)^{-2}$, which is why we call it the improved quasi-geostrophic potential vorticity (IQGPV) equation.

The proof of these results with detailed calculations are available as a supplement to the online version, or from the authors or the Journal of Fluid Mechanics Editorial Office, Cambridge. The calculations use the decomposition of the velocity and the pressure fields into barotropic and baroclinic components and follow the corresponding calculations of Part 1.

2.3. *The FG regime*

2.3.1. *Definitions and the basic equations*

The FG regime corresponds to interface displacements of the order one. The FG scaling, thus, differs from the QG one used before. It is as follows. The interface displacement η is scaled as H_1. Choosing the characteristic length scale L_0 we rescale the pressure perturbations π_i (cf. (2.3)) by $\rho_i f_0 V_i L_0$, where V_i, $i = 1, 2$ are the velocity scales in each layer. The velocity scales V_1 and V_2 are of the same order when the parameter d is of order one, which corresponds to the FGH sub-regime (cf. Benilov & Reznik 1996) or are chosen to be $V_2 \sim \epsilon V_1$ for the FGI sub-regime where the parameter d is small, $d \sim \epsilon^2$ (cf. Cushman-Roisin *et al.* 1992). The consistency of these scalings with the dynamical boundary condition on the interface (2.5) requires that in order to have order-one (frontal) interface displacements the Burger number $Bu = (R_R/L_0)^2$ should be small $Bu = O(\epsilon)$. Here the Rossby deformation radius is defined with the help of H_1: $R_R = \sqrt{g' H_1}/f_0$.

Introducing the complex variables $\xi = x + \mathrm{i}y$, $\xi^* = x - \mathrm{i}y$ (cf. Part 1) we obtain the following non-dimensional equations for the FG regime:

$$\left.\begin{aligned}
&\partial_t \mathscr{U}_i + \mathrm{i}\mathscr{U}_i + \epsilon(\mathscr{U}_i\partial_\xi\mathscr{U}_i + \mathscr{U}_i^*\partial_{\xi^*}\mathscr{U}_i) = -2\partial_{\xi^*}\pi_i, \quad i = 1, 2,\\
&\partial_\xi[(1-\eta)\mathscr{U}_1 + (d^{-1}+\eta)\mathscr{U}_2] + \text{c.c.} = 0,\\
&\partial_t\eta = \epsilon\partial_\xi[(1-\eta)\mathscr{U}_1] + \text{c.c.},\\
&\pi_2 = \pi_1 + \eta,
\end{aligned}\right\} \tag{2.18}$$

where $\mathscr{U}_{1,2} = u_{1,2} + \mathrm{i}v_{1,2}$ are the complex velocities in respective layers and η is the interface displacement.

It is convenient to rewrite (2.18) in terms of the barotropic and the baroclinic modes (cf. Benilov & Reznik 1996):

$$\partial_t \mathscr{U}_{bt} + \mathrm{i}\mathscr{U}_{bt} + \frac{\epsilon d}{1+d}\left[\partial_\xi\left(\mathscr{U}_{bt}^2 + \Phi\mathscr{U}_{bc}^2\right) + \partial_{\xi^*}(|\mathscr{U}_{bt}|^2 + \Phi|\mathscr{U}_{bc}|^2)\right] = -2\partial_{\xi^*}P, \quad (2.19)$$

$$\begin{aligned}\partial_t \mathscr{U}_{bc} + \mathrm{i}\mathscr{U}_{bc} + \frac{\epsilon d}{1+d}&[\partial_\xi(\mathscr{U}_{bt}\mathscr{U}_{bc}) + \mathscr{U}_{bc}^*\partial_{\xi^*}\mathscr{U}_{bt} + \mathscr{U}_{bt}^*\partial_{\xi^*}\mathscr{U}_{bc}\\ &+\mathscr{U}_{bc}[\partial_\xi((d^{-1}+\eta) - \mathscr{U}_{bc}) - (1-\eta)\partial_\xi\mathscr{U}_{bc}]\\ &+\mathscr{U}_{bc}^*[\partial_{\xi^*}((d^{-1}+\eta)\mathscr{U}_{bc}) - (1-\eta)\partial_{\xi^*}\mathscr{U}_{bc}]] = 2\partial_{\xi^*}\eta, \end{aligned} \quad (2.20)$$

$$\partial_\xi \mathscr{U}_{bt} + \text{c.c.} = 0, \quad (2.21)$$

$$\partial_t \eta = \frac{\epsilon d}{1+d}[-\mathscr{U}_{bt}\partial_\xi\eta + \partial_\xi(\Phi\mathscr{U}_{bc})] + \text{c.c.} \quad (2.22)$$

Here

$$\mathscr{U}_{bt} = (1-\eta)\mathscr{U}_1 + (d^{-1}+\eta)\mathscr{U}_2, \quad \mathscr{U}_{bc} = \mathscr{U}_1 - \mathscr{U}_2, \quad P = \pi_1 + d^{-1}\pi_2 + \frac{\eta^2}{2} \quad (2.23)$$

and the notation $\Phi = (1-\eta)(d^{-1}+\eta)$ is used for compactness.

2.3.2. *The main results*

As in the QG regime all fields are split into slow and fast parts (cf. (2.13)). Both the slow and the fast components evolve from uniquely defined initial conditions. However, unlike the QG case, the fast component consists not of the propagating IGW but of inertial oscillations with slowly changing amplitude. For example, the barotropic and baroclinic complex velocities in both sub-regimes are expressed as follows:

$$\mathscr{U}_{bt} = 2\mathrm{i}\partial_{\xi^*}P, \quad \mathscr{U}_{bc} = -2\mathrm{i}\partial_{\xi^*}\eta + \mathscr{A}\mathrm{e}^{-\mathrm{i}t}. \quad (2.24)$$

Here the slow functions η, P denote the leading-order interface displacement and the barotropic pressure, respectively, and $\mathscr{A} = \mathscr{A}(\xi, \xi^*, t_1, \ldots)$ is the slowly evolving envelope of the inertial oscillations. Correspondingly, the leading-order evolution is determined by two coupled equations for slow P and η and a separate equation for $\mathscr{A}$. The evolution equations obtained below are as follows:

FGH sub-regime

$$\partial_{t_1}\eta = \frac{1}{1+d^{-1}}J(\eta, P), \quad (2.25)$$

$$\partial_{t_1}\nabla^2 P + \frac{1}{1+d^{-1}}[J(P, \nabla^2 P) + \nabla\cdot((1-\eta)(d^{-1}+\eta)J(\eta, \nabla\eta))] = 0, \quad (2.26)$$

$$\begin{aligned}&(1+d^{-1})\partial_{t_1}\mathscr{A} + J(P-\eta^2-(d^{-1}-1)\eta, \mathscr{A})\\ &\quad+\frac{\mathrm{i}}{2}[\nabla^2(P-\eta^2-(d^{-1}-1)\eta)+(\nabla\eta)^2]\mathscr{A} - \frac{\mathrm{i}}{2}\nabla^2((1-\eta)(d^{-1}+\eta)\mathscr{A}) = 0; \end{aligned} \quad (2.27)$$

FGI sub-regime

$$\partial_{t_2}\eta + J(P, \eta) + J\left(\eta, (1-\eta)\nabla^2\eta - \frac{(\nabla\eta)^2}{2}\right) = 0, \quad (2.28)$$

$$\partial_{t_2}\nabla^2 P + J(P, \nabla^2 P) - J\left(\eta, (1-\eta)\nabla^2\eta + \frac{(\nabla\eta)^2}{2}\right) = 0, \quad (2.29)$$

$$\partial_{t_1}\mathscr{A} - J(\eta, \mathscr{A}) - \frac{\mathrm{i}}{2}\nabla^2\mathscr{A} + \frac{\mathrm{i}}{2}[\nabla^2(\eta\mathscr{A}) - \mathscr{A}\nabla^2\eta] = 0. \quad (2.30)$$

The most important and non-trivial feature of both FG sub-regimes is that although the inertial oscillations do not run away like the IGW in the preceding Section, they do not make any contribution to the evolution of the slow component. In other words, the fast oscillations exert no drag on the slow vortical motion. At the same time, the slow modulation of the inertial oscillations is guided by the vortical motion since the coefficients in the modulation equations (2.27), (2.30) depend on P, η.

Another interesting point is the difference between the FGH and the FGI sub-regimes. In the former the slow component and the modulation amplitude evolve in the same slow time t_1, while in the latter the slow component evolves in the slow time t_2, and the amplitude $\mathscr{A}$ in the faster time t_1. This is a novel feature arising due to stratification: in the barotropic RSW model the single FG regime is analogous to the FGI sub-regime (see Part 1 for details).

2.3.3. *FGH: calculations at the lowest and the first order*

The momentum and the continuity equations give at the lowest order:

$$\left.\begin{aligned} \partial_t \mathscr{U}_{bt}^{(0)} + \mathrm{i}\mathscr{U}_{bt}^{(0)} &= -2\partial_{\xi^*} P^{(0)}, \\ \partial_t \mathscr{U}_{bc}^{(0)} + \mathrm{i}\mathscr{U}_{bc}^{(0)} &= 2\partial_{\xi^*}\eta^{(0)}, \end{aligned}\right\} \tag{2.31}$$

$$\partial_\xi \mathscr{U}_{bt}^{(0)} + \text{c.c.} = 0, \quad \partial_t \eta^{(0)} = 0. \tag{2.32}$$

Hence

$$\mathscr{U}_{bt}^{(0)} = 2\mathrm{i}\partial_{\xi^*} P^{(0)}, \quad P^{(0)} = P^{(0)}(x, y, t_1, \ldots), \tag{2.33}$$

$$\mathscr{U}_{bc}^{(0)} = \bar{\mathscr{U}}_{bc}^{(0)} + \tilde{\mathscr{U}}_{bc}^{(0)} = -2\mathrm{i}\partial_{\xi^*}\eta^{(0)} + \mathscr{A}^{(0)}\mathrm{e}^{-\mathrm{i}t}, \quad \eta^{(0)} = \eta^{(0)}(x, y, t_1, \ldots). \tag{2.34}$$

Proper initial conditions for the slow and the fast parts readily follow from the above equations. Note however that in order to respect the FG scaling the barotropic component of the initial velocity should be almost divergenceless (the same is true in the FGI case below).

At the first order we have

$$\left.\begin{aligned} \partial_t \mathscr{U}_{bt}^{(1)} + \mathrm{i}\mathscr{U}_{bt}^{(1)} &= -2\partial_{\xi^*} P^{(1)} + F_{bt}^{(1)}, \\ \partial_t \mathscr{U}_{bc}^{(1)} + \mathrm{i}\mathscr{U}_{bc}^{(1)} &= 2\partial_{\xi^*}\eta^{(1)} + F_{bc}^{(1)}, \end{aligned}\right\} \tag{2.35}$$

$$\partial_\xi \mathscr{U}_{bt}^{(1)} + \text{c.c.} = 0, \tag{2.36}$$

$$\partial_t \eta^{(1)} + \partial_{t_1}\eta^{(0)} = \frac{1}{1+d^{-1}}\left[-\mathscr{U}_{bt}^{(0)}\partial_\xi \eta^{(0)} + \partial_\xi\left(\Phi^{(0)}\mathscr{U}_{bc}^{(0)}\right)\right] + \text{c.c.}, \tag{2.37}$$

where we define

$$F_{bt}^{(1)} = -\partial_{t_1}\mathscr{U}_{bt}^{(0)} - \frac{1}{1+d^{-1}}\left[\partial_\xi\left(\mathscr{U}_{bt}^{(0)2} + \Phi^{(0)}\mathscr{U}_{bc}^{(0)2}\right) + \partial_{\xi^*}\left(\left|\mathscr{U}_{bt}^{(0)}\right|^2 + \Phi^{(0)}\left|\mathscr{U}_{bc}^{(0)}\right|^2\right)\right], \tag{2.38}$$

$$\begin{aligned} F_{bc}^{(1)} = -\partial_{t_1}\mathscr{U}_{bc}^{(0)} - \frac{1}{1+d^{-1}}\Big[&\partial_\xi\left(\mathscr{U}_{bt}^{(0)}\mathscr{U}_{bc}^{(0)}\right) + \mathscr{U}_{bc}^{(0)*}\partial_{\xi^*}\mathscr{U}_{bt}^{(0)} + \mathscr{U}_{bt}^{(0)*}\partial_{\xi^*}\mathscr{U}_{bc}^{(0)} \\ &+ \mathscr{U}_{bc}^{(0)}\left[\partial_\xi\left(\left(d^{-1}+\eta^{(0)}\right)\mathscr{U}_{bc}^{(0)}\right) - \left(1-\eta^{(0)}\right)\partial_\xi \mathscr{U}_{bc}^{(0)}\right] \\ &+ \mathscr{U}_{bc}^{(0)*}\left[\partial_{\xi^*}\left(\left(d^{-1}+\eta^{(0)}\right)\mathscr{U}_{bc}^{(0)}\right) - \left(1-\eta^{(0)}\right)\partial_{\xi^*}\mathscr{U}_{bc}^{(0)}\right]\Big]. \end{aligned} \tag{2.39}$$

From (2.37) and (2.33) the equation (2.25) for $\eta^{(0)}$ follows together with

$$\eta^{(1)} = \frac{1}{1+d^{-1}}\mathrm{i}\mathrm{e}^{-\mathrm{i}t}\partial_\xi\left(\Phi^{(0)}\mathscr{A}^{(0)}\right) + \text{c.c.} + \bar{\eta}^{(1)}. \tag{2.40}$$

From (2.35), (2.36), (2.33) we obtain

$$\partial_\xi \bar{F}^{(1)}_{bt} - \partial_{\xi^*} \bar{F}^{(1)^*}_{bt} = 0, \tag{2.41}$$

$$\bar{F}^{(1)}_{bt} = -\partial_{t_1} \mathscr{U}^{(0)}_{bt} - \frac{1}{1+d^{-1}} \left[\partial_\xi \left(\mathscr{U}^{(0)2}_{bt} + \Phi^{(0)} \bar{\mathscr{U}}^{(0)^2}_{bc} \right) + \partial_{\xi^*} \left\langle \left| \mathscr{U}^{(0)}_{bt} \right|^2 + \Phi^{(0)} \left| \mathscr{U}^{(0)}_{bc} \right|^2 \right\rangle \right]. \tag{2.42}$$

From (2.41), (2.42), (2.33), (2.34) the evolution equation (2.26) for the barotropic pressure component $P^{(0)}$ is obtained. It is worth noting that $\mathscr{A}^{(0)}$ enters in the equation for $\bar{F}^{(1)}_{bt}$ but cancels out in the equation for $P^{(0)}$.

By virtue of (2.40) the right-hand side of the second equation in (2.35) is well-defined and the condition of absence of secular terms gives the equation (2.27) for the amplitude $\mathscr{A}^{(0)}$.

2.3.4. *FGI: preliminaries*

In the FGI sub-regime the scaling should be changed because of the shallow upper layer. Hence, we take (cf. Cushman-Roisin *et al.* 1992)

$$d = O(\epsilon^2), \quad \mathscr{U}_1 = O(1), \quad \mathscr{U}_2 = O(\epsilon) \tag{2.43}$$

and therefore

$$\mathscr{U}_{bt} = (1-\eta)\mathscr{U}_1 + (d^{-1} + \eta)\mathscr{U}_2 = O(\epsilon^{-1}), \quad \mathscr{U}_{bc} = \mathscr{U}_1 - \mathscr{U}_2 = O(1). \tag{2.44}$$

Correspondingly, a solution is sought in the form

$$\left.\begin{aligned} \mathscr{U}_{bc} &= \mathscr{U}^{(0)}_{bc} + \epsilon \mathscr{U}^{(1)}_{bc} + \epsilon^2 \mathscr{U}^{(2)}_{bc} + \ldots, \\ \mathscr{U}_{bt} &= \epsilon^{-1} \mathscr{U}^{(0)}_{bc} + \mathscr{U}^{(1)}_{bc} + \epsilon \mathscr{U}^{(2)}_{bc} + \ldots, \\ P &= \epsilon^{-1} P^{(0)} + P^{(1)} + \epsilon P^{(2)} + \ldots, \\ \eta &= \eta^{(0)} + \epsilon \eta^{(1)} + \epsilon^2 \eta^{(2)} + \ldots. \end{aligned}\right\} \tag{2.45}$$

In what follows we assume that $d = \epsilon^2$ for simplicity of notation.

2.3.5. *FGI: calculations*

The lowest-order calculation coincides exactly with the FGH one and equations (2.31)–(2.32) remain valid. At the first order we have

$$\partial_t \mathscr{U}^{(1)}_{bt} + \mathrm{i} \mathscr{U}^{(1)}_{bt} = -2\partial_{\xi^*} P^{(1)} - \partial_{t_1} \mathscr{U}^{(0)}_{bt}, \tag{2.46}$$

$$\partial_t \mathscr{U}^{(1)}_{bc} + \mathrm{i} \mathscr{U}^{(1)}_{bc} = 2\partial_{\xi^*} \eta^{(1)} - \partial_{t_1} \mathscr{U}^{(0)}_{bc} - \mathscr{U}^{(0)}_{bc} \partial_\xi \mathscr{U}^{(0)}_{bc} - \mathscr{U}^{(0)*}_{bc} \partial_{\xi^*} \mathscr{U}^{(0)}_{bc}, \tag{2.47}$$

$$\partial_\xi \mathscr{U}^{(1)}_{bt} + \text{c.c.} = 0, \tag{2.48}$$

$$\partial_t \eta^{(1)} + \partial_{t_1} \eta^{(0)} = \partial_\xi \left[\left(1 - \eta^{(0)}\right) \mathscr{U}^{(0)}_{bc} \right] + \text{c.c.} \tag{2.49}$$

Like in the frontal regime in Part 1, by removing resonances we obtain the following modulation equation for the amplitude of inertial oscillations:

$$\begin{aligned} -\partial_{t_1} \mathscr{A}^{(0)} + 2\mathrm{i}\partial^2_{\xi\xi^*} \left(\left(1 - \eta^{(0)}\right) \mathscr{A}^{(0)}_1 \right) + 2\mathrm{i} \mathscr{A}^{(0)} \partial^2_{\xi\xi^*} \eta^{(0)} \\ + 2\mathrm{i} \left(\partial_{\xi^*} \eta^{(0)} \partial_\xi \mathscr{A}^{(0)} - \partial_\xi \eta^{(0)} \partial_{\xi^*} \mathscr{A}^{(0)} \right) = 0. \end{aligned} \tag{2.50}$$

In the real notation this equation takes the form (2.30).

After elimination of the secular growth, the regular solution of the fast part of (2.47) may be easily found (see an analogous calculation in Part 1). From (2.46), (2.48), and (2.33) we obtain the following equations for the slow barotropic variables:

$$\mathscr{U}_{bt}^{(1)} = 2i\partial_{\xi^*} P^{(1)}, \quad P^{(1)} = P^{(1)}(x, y, t_1, \ldots), \quad P^{(0)} = P^{(0)}(x, y, t_2, \ldots). \tag{2.51}$$

The first slow correction to the baroclinic velocity field is obtained by averaging of (2.47)

$$\bar{\mathscr{U}}_{bc}^{(1)} = -\mathrm{i}\big(2\partial_{\xi^*}\bar{\eta}^{(1)} + 4\big(\partial_{\xi^*}\bar{\eta}^{(0)}\partial^2_{\xi\xi^*}\bar{\eta}^{(0)} - \partial_{\xi}\bar{\eta}^{(0)}\partial^2_{\xi^*\xi^*}\bar{\eta}^{(0)}\big) - \mathscr{A}^{(0)*}\partial_{\xi^*}\mathscr{A}^{(0)}\big). \tag{2.52}$$

Again, note that the amplitude of inertial oscillations enters this expression. At second order the momentum equations give

$$\partial_t \mathscr{U}_{bt}^{(2)} + \mathrm{i}\mathscr{U}_{bt}^{(2)} = -2\partial_{\xi^*} P^{(2)} + F_{bt}^{(2)}, \tag{2.53}$$

$$\partial_t \mathscr{U}_{bc}^{(2)} + \mathrm{i}\mathscr{U}_{bc}^{(2)} = 2\partial_{\xi^*}\eta^{(2)} + F_{bc}^{(2)}. \tag{2.54}$$

Here

$$F_{bt}^{(2)} = -\partial_{t_2}\mathscr{U}_{bt}^{(0)} - \partial_{t_1}\mathscr{U}_{bt}^{(1)} - \partial_{\xi}\big[\big(\mathscr{U}_{bt}^{(0)}\big)^2 + \big(1-\eta^{(0)}\big)\big(\mathscr{U}_{bc}^{(0)}\big)^2\big] \\ - \partial_{\xi^*}\big[\big|\mathscr{U}_{bt}^{(0)}\big|^2 + \big(1-\eta^{(0)}\big)\big|\mathscr{U}_{bc}^{(0)}\big|^2\big] \tag{2.55}$$

and

$$F_{bc}^{(2)} = -\partial_{t_2}\mathscr{U}_{bc}^{(0)} - \partial_{t_1}\mathscr{U}_{bc}^{(1)} - \partial_{\xi}\big(\mathscr{U}_{bt}^{(0)}\mathscr{U}_{bc}^{(0)}\big) - \mathscr{U}_{bc}^{(0)*}\partial_{\xi^*}\mathscr{U}_{bt}^{(0)} - \mathscr{U}_{bt}^{(0)*}\partial_{\xi^*}\mathscr{U}_{bc}^{(0)} \\ - \mathscr{U}_{bc}^{(1)}\partial_{\xi}\mathscr{U}_{bc}^{(0)} - \mathscr{U}_{bc}^{(1)*}\partial_{\xi^*}\mathscr{U}_{bc}^{(0)} - \mathscr{U}_{bc}^{(0)}\big[\partial_{\xi}\mathscr{U}_{bc}^{(1)} - \big(1-\eta^{(0)}\big)\partial_{\xi}\mathscr{U}_{bc}^{(0)}\big] \\ - \mathscr{U}_{bc}^{(0)*}\big[\partial_{\xi^*}\mathscr{U}_{bc}^{(1)} - \big(1-\eta^{(0)}\big)\partial_{\xi^*}\mathscr{U}_{bc}^{(0)}\big]. \tag{2.56}$$

The divergence and mass-conservation equations give, respectively,

$$\partial_{\xi}\mathscr{U}_{bt}^{(2)} + \text{c.c.} = 0, \tag{2.57}$$

$$\partial_{t_2}\eta^{(0)} + \partial_{t_1}\eta^{(1)} + \partial_t\eta^{(2)} - \partial_{\xi}\big(\big(1-\eta^{(0)}\big)\mathscr{U}_{bc}^{(1)} - \eta^{(1)}\mathscr{U}_{bc}^{(0)}\big) + \mathscr{U}_{bt}^{(0)}\partial_{\xi}\eta^{(0)} + \text{c.c.} = 0. \tag{2.58}$$

By averaging this equation in t and supposing that $\eta^{(2)}$ is bounded in time we obtain

$$\partial_{t_1}\bar{\eta}^{(1)} + \partial_{t_2}\bar{\eta}^{(0)} - \big(\partial_{\xi}\big[\big(1-\eta^{(0)}\big)\bar{\mathscr{U}}_{bc}^{(1)}\big] - \partial_{\xi}\big(\eta^{(1)}\bar{\mathscr{U}}_{bc}^{(0)}\big) + \text{c.c.}\big) \\ + \big\langle\partial_{\xi}\big(\tilde{\eta}^{(1)}\tilde{\mathscr{U}}_{bc}^{(0)}\big) + \text{c.c.}\big\rangle - \big(2i\partial_{\xi^*}P^{(0)}\partial_{\xi}\eta^{(0)} + \text{c.c.}\big) = 0. \tag{2.59}$$

The evolution equation for $\bar{\eta}^{(0)}$ follows from (2.59) as a condition of absence of secular growth of $\bar{\eta}^{(1)}$ in t_1. Coming back to the real notation we obtain (2.28) for $\bar{\eta}^{(0)}$.

From (2.53), (2.57) we have

$$\partial_t\big(\partial_{\xi}\mathscr{U}_{bt}^{(2)} - \text{c.c.}\big) = -\mathrm{i}\big(\partial_{t_2}\nabla^2 P^{(0)} + \partial_{t_1}\nabla^2 P^{(1)}\big) + \partial_{\xi}F_{bt}^{(2)} - \text{c.c.} = 0. \tag{2.60}$$

By time averaging this equation and using (2.33), (2.34), (2.51), (2.55) we find the evolution equation (2.29) for $P^{(0)}$.

A calculation analogous to that made in Part 1 shows that there are no nonlinear (cubic) corrections to this Schrödinger equation at the next order of the perturbation theory. Note also that, although we worked with the non-filtered equation in both FGH and FGI cases we could not avoid a self-consistency constraint of balanced initial conditions, namely that the barotropic component of the initial velocity field should be almost divergenceless.

3. Nonlinear geostrophic adjustment in a continuously stratified model

3.1. *Preliminaries*

The hydrostatic primitive equations in the Boussinesq approximation (HSPE) can be written in standard notations as follows:

$$\left.\begin{aligned} &\partial_t \boldsymbol{v}_h + \boldsymbol{v}\cdot\nabla\boldsymbol{v}_h + f_0\hat{\boldsymbol{z}}\wedge\boldsymbol{v}_h + \frac{1}{\rho_0}\nabla_h P = 0,\\ &\partial_z P + \rho g = 0,\\ &\partial_t\rho + \boldsymbol{v}\cdot\nabla\rho = 0,\\ &\nabla\cdot\boldsymbol{v} = 0. \end{aligned}\right\} \tag{3.1}$$

Here the subscript h denotes the horizontal part of the fields (which depend now on the full $\boldsymbol{r}=(x, y, z)$) or operators. The velocity field is now three-dimensional: $\boldsymbol{v}=(\boldsymbol{v}_h, w)$, as is nabla $\nabla=(\nabla_h, \partial_z)$ and we close the system by requiring that the vertical velocity vanishes at the top and the bottom boundaries:

$$w|_{z=-H} = w|_{z=0} = 0. \tag{3.2}$$

The density ρ and the pressure P may be decomposed into static and dynamic parts (we consider the oceanographic context with $\rho_0 \gg \rho_s$; the calculations may be repeated with corresponding changes for the atmospheric context, see, e.g. Vallis 1996 for a description of the balanced motion):

$$\rho = \rho_0 + \rho_s(z) + \lambda\rho'(x, y, z; t), \quad P = p_s(z) + \lambda p'(x, y, z; t), \tag{3.3}$$

where

$$p_s = -\rho_0 g z + g\int_z^H \rho_s(z', t)\,\mathrm{d}z' \tag{3.4}$$

and λ is a non-dimensional amplitude of the relative deviations of the isopycnal surfaces. Throughout this section ρ_s is assumed to be a stably stratified density profile and primes in the dynamical part of the density and pressure will be omitted.

We solve an initial-value problem with initial conditions

$$\boldsymbol{v}_h|_{t=0} = \boldsymbol{v}_{h_I}(x, y, z), \quad \rho|_{t=0} = \rho_I(x, y, z) \tag{3.5}$$

(pressure and vertical velocity are not independent variables in HSPE and may be expressed in terms of other variables via hydrostatics and incompressibility equations in (3.1)).

To be consistent with the Boussinesq approximation (divergenceless of velocity) and boundary conditions (3.2) the initial horizontal divergence $D_I = \nabla_h\cdot\boldsymbol{v}_h$ should obey the following relation:

$$\int_{-H}^{0} \mathrm{d}z\, D_I = 0. \tag{3.6}$$

The PV equation has the form

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\Pi = 0, \tag{3.7}$$

where

$$\Pi = (\boldsymbol{\omega} + \hat{\boldsymbol{z}} f_0)\cdot\nabla(\rho_s(z) + \lambda\rho) \tag{3.8}$$

and

$$\boldsymbol{\omega} = (-\partial_z v, \partial_z u, \partial_x v - \partial_y u) \tag{3.9}$$

is the three-dimensional relative vorticity in the hydrostatic approximation. Introducing, as usual, the characteristic horizontal scale L and the characteristic horizontal velocity scale U we define the Rossby number $\epsilon = U/f_0 L$. The characteristic vertical scale is the fluid layer thickness, $H \ll L$, and, to be consistent with incompressibility, the vertical velocity scale is $W \sim UH/L$. The characteristic pressure scale is $\rho_0 f_0 U L$ and the characteristic density variations scale is $\rho_0 f_0 U L/gH$ as dictated by the Boussinesq equations (3.1). The non-dimensional version of (3.1) is

$$\left.\begin{aligned} &\partial_t \boldsymbol{v}_h + \epsilon \boldsymbol{v}\cdot\nabla\boldsymbol{v}_h + \hat{\boldsymbol{z}}\wedge\boldsymbol{v}_h + \nabla_h p = 0,\\ &\partial_z p + \rho = 0,\\ &\partial_t \rho + \epsilon\boldsymbol{v}\cdot\nabla\rho - sN^2 w = 0,\\ &\nabla\cdot\boldsymbol{v} = 0, \end{aligned}\right\} \tag{3.10}$$

where the Burger number $Bu \equiv s = R_d^2/L^2$ and the baroclinic Rossby radius $R_d = N_0 H/f_0$ are introduced with N_0 denoting the characteristic scale of the Brunt–Väisälä frequency $N = (-(g/\rho_0)(\mathrm{d}\rho_s(z)/\mathrm{d}z)^{1/2}$. We impose, as usual, (cf. e.g. Romanova & Zeitlin 1984) the quasi-geostrophy condition $\lambda Bu/Ro = O(1)$ and omit all order-one numerical factors.

The PV equation can be written as

$$(\partial_t + \epsilon\boldsymbol{v}\cdot\nabla)\Pi = 0, \quad \Pi = \Pi_0 + \epsilon\Pi_1 + \epsilon^2\Pi_2, \tag{3.11}$$

where

$$\Pi_0 = -sN^2, \quad \Pi_1 = -s\zeta N^2 + \partial_z\rho, \quad \Pi_2 = -\partial_z v\partial_x\rho + \partial_z u\partial_y\rho + \zeta\partial_z\rho, \tag{3.12}$$

and ζ denotes the vertical component of the relative vorticity $\partial_x v - \partial_y u$, as usual. Using the density advection equation in (3.10) one can rewrite (3.11) as follows:

$$\partial_t\left(N^2\Omega - \epsilon\Pi_2 - \frac{\epsilon}{s}\sigma\rho^2\right) + \epsilon\boldsymbol{v}\cdot\nabla\left(N^2\Omega - \epsilon\Pi_2 - \frac{\epsilon}{s}\sigma\rho^2\right) + \frac{\epsilon^2}{s}\frac{\mathrm{d}\sigma}{\mathrm{d}z}w\rho^2 = 0, \tag{3.13}$$

where we denote, for brevity

$$\Omega = s\zeta - \partial_z\left(\frac{\rho}{N^2}\right), \quad \sigma = \frac{1}{2N^2}\frac{\mathrm{d}^2\log N^2}{\mathrm{d}z^2}. \tag{3.14}$$

3.2. *The QG regime*

3.2.1. *Definitions and statement of the main results*

In this Section we assume that the relative deviations of the isopycnal surfaces are small, $\lambda = O(\epsilon)$. Then from the quasi-geostrophy condition $\lambda s/\epsilon = O(1)$ it follows that $s = O(1)$, which means that the scale of the motion is of the order of the Rossby deformation radius, $L \sim R_d$.

Qualitatively, the evolution of an arbitrary initial perturbation with small Rossby number and Burger number of order unity is analogous to the two-layer QG case considered in § 2.2, although the mathematics is more complicated. Again, our analysis is performed up to the third order in ϵ. All fields are split in a unique way into non-interacting slow and fast components obeying their proper evolution equations with uniquely defined initial conditions.

The slow component obeys the single 'improved' QG equation for the slow pressure field $\bar{p} = \bar{p}^{(0)} + \epsilon\bar{p}^{(1)}$:

$$\frac{\mathrm{D}}{\mathrm{D}t_1}\left[\partial_z\left(\frac{1}{N^2}\partial_z\bar{p}\right) + \nabla_h^2\bar{p} - \epsilon 2J(\partial_x\bar{p}, \partial_y\bar{p}) + \frac{\epsilon}{N^2}\left(\partial_{zz}^2\bar{p}\nabla_h^2\bar{p} - \left(\partial_{zx}^2\bar{p}\right)^2 - \left(\partial_{zy}^2\bar{p}\right)^2 \right.\right.$$
$$\left.\left. - \sigma(\partial_z\bar{p})^2 - \nabla_N\bar{p}\cdot\nabla\left[N^2\left(\nabla_h^2\bar{p} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}\right)\right)\right]\right)\right] = 0, \tag{3.15}$$

where $\mathrm{D}/\mathrm{D}t_1$ is the following advective derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t_1}\ldots=\partial_{t_1}\ldots+J\left(\bar{p}-\frac{\epsilon}{2}\nabla\bar{p}\cdot\nabla_N\bar{p},\ldots\right) \tag{3.16}$$

and $\nabla_N=(\nabla_h,\partial_z/N^2)$. Equation (3.15) describes the QG motion on time scales of the order of $\epsilon^{-2}f^{-1}$, i.e. on much longer time scales than the standard QG equation which follows from (3.15) if the $O(\epsilon)$ terms are neglected.

The fast component consists of the internal IGW emitted by the localized initial disturbance. The waves obey the inhomogeneous wave equation for the fast pressure field $\tilde{p}=\tilde{p}^{(0)}+\epsilon\tilde{p}^{(1)}$:

$$\partial^2_{tt}\partial_z\left(\frac{1}{N^2}\partial_z\tilde{p}\right)+\partial_z\left(\frac{1}{N^2}\partial_z\tilde{p}\right)+\nabla^2_h\tilde{p}=\epsilon\mathscr{R}(x,y,z;t,t_1,\ldots), \tag{3.17}$$

with the boundary conditions

$$\partial_z\tilde{p}|_{z=-1,0}=0 \tag{3.18}$$

and well-defined initial conditions. The known right-hand side in (3.17) results from nonlinear interactions of the lowest-order fast component with itself and with the slow one. As in the 2RSW case, these interactions produce no significant resonances and, therefore, the fast fields decay in time at a fixed spatial location and induce no drag in the slow equation (3.15).

The full demonstration is available as a supplement to the online version or on request from the authors or the Journal of Fluid Mechanics Editorial Office, Cambridge. It generally follows that of Part 1 except for two new technical ingredients: the decomposition of the fast part of the pressure with the help of the eigenfunctions of the eigenproblem

$$\partial_z\left(\frac{1}{N^2}\partial_z\Psi_m\right)+\lambda^2_m\Psi_m=0;\quad \partial_z\Psi_m|_{z=-1,0}=0,\quad m=0,1,\ldots, \tag{3.19}$$

and the use of the PV-equation in the form (3.13).

3.3. *The FG regime*

3.3.1. *Preliminaries*

The stratified FG regime, like the two-layer ones considered above, is characterized by $O(1)$ isopycnal deviations, i.e. the parameter $\lambda=O(1)$ and, hence, $Bu=O(\epsilon)\ll 1$. Therefore we do not use here the representation (3.3) for the density variable ρ; the non-dimensional density equation in (3.10) is substituted by the following one:

$$\partial_t\rho+\epsilon\boldsymbol{v}\cdot\nabla\rho=0. \tag{3.20}$$

As usual in the frontal regime, it is convenient to introduce the complex horizontal coordinates ξ,ξ^* and the complex velocity $\mathscr{U}$. Taking into account (3.20) the system (3.10) can then be rewritten as

$$\left.\begin{array}{l}\partial_t\mathscr{U}+\mathrm{i}\mathscr{U}+2\partial_{\xi^*}p+\epsilon[(\mathscr{U}\partial_\xi+\mathscr{U}^*\partial_{\xi^*})\mathscr{U}+w\partial_z\mathscr{U}]=0,\\ \partial_z p+\rho=0,\\ \partial_t\rho+\epsilon[(\mathscr{U}\partial_\xi+\mathscr{U}^*\partial_{\xi^*})\rho+w\partial_z\rho]=0,\\ \partial_\xi\mathscr{U}+\partial_{\xi^*}\mathscr{U}^*+\partial_z w=0.\end{array}\right\} \tag{3.21}$$

The initial conditions are

$$(\mathscr{U},\rho)_{t=0}=(\mathscr{U}_I,\rho_I) \tag{3.22}$$

and we use, as before, the rigid lid top and bottom boundary conditions:

$$w|_{z=-1,0} = 0. \tag{3.23}$$

Hence, by virtue of the continuity equation in (3.21) the barotropic part of the initial horizontal velocity field $\mathscr{U}_{bt_I} = \int_{-1}^{0} \mathrm{d}z\, \mathscr{U}_I$ is divergenceless:

$$\partial_\xi \mathscr{U}_{bt_I} + \partial_{\xi^*} \mathscr{U}^*_{bt_I} = 0. \tag{3.24}$$

3.3.2. *The statement of the main results*

The FG regime which we consider here is similar to the FGH sub-regime in the two-layer model. The lowest-order velocity field is represented in the form

$$\mathscr{U}^{(0)} = \mathscr{A}(x, y, z; t_1)\,\mathrm{e}^{-\mathrm{i}t} + 2\mathrm{i}\partial_{\xi^*}\bar{p}, \quad w = -\partial_{\xi^*}\mathscr{M}\mathrm{e}^{-\mathrm{i}t} + \text{c.c.}, \tag{3.25}$$

where the slow modulation amplitude $\mathscr{A}$ of the inertial oscillations and the function $\mathscr{M}$ are related as follows:

$$\partial_z \mathscr{M} = \mathscr{A}. \tag{3.26}$$

As we see from (3.25), only the horizontal velocity $\mathscr{U}$ contains a slow component expressed in terms of the slow pressure $\bar{p}$; the lowest-order vertical velocity w is due to inertial oscillations only.

The slow pressure obeys the following pair of equations:

$$\int_{-1}^{0} \mathrm{d}z\left[\partial_{t_1}\nabla^2\overline{p} + J(\overline{p}, \nabla^2\overline{p})\right] = 0, \tag{3.27}$$

$$\partial^2_{t_1 z}\overline{p} + J(\overline{p}, \partial_z\overline{p}) = 0, \tag{3.28}$$

which were first introduced by Benilov (1993). Evolution of the fast oscillations is conveniently described by the modulation equation for $\mathscr{M}$:

$$\partial_{t_1}\partial^2_{zz}\mathscr{M} + J\left(\overline{p}, \partial^2_{zz}\mathscr{M}\right) + \mathrm{i}\frac{\nabla^2_h\overline{p}}{2}\partial^2_{zz}\mathscr{M} - \frac{\mathrm{i}}{2}(\partial_z\overline{\rho})\nabla^2_h\mathscr{M} - \mathrm{i}[\partial_z(\partial_x + \mathrm{i}\partial_y)\overline{p}][\partial_z(\partial_x - \mathrm{i}\partial_y)\mathscr{M}] = 0. \tag{3.29}$$

Again, we see that the slow component does not ‘feel’ the fast oscillations which means no drag terms in (3.27), (3.28). At the same time the coefficients of the modulation equation (3.29) depend on the slow pressure and density fields, i.e. the fast component is guided by the slow one as in the FG regimes considered above. An important point is that the inertial oscillations envelope and the slow vortex field evolve in the same time t_1, like in the two-layer FGH regime, and there is no time-scale separation between the two, unlike the RSW FG regime and 2RSW FGI regime. As the initial conditions considered below are almost arbitrary, a natural question arises: is an analogue of the two-layer FGI regime possible in the continuously stratified model? We address this point in §4.

3.3.3. *The lowest-order solution*

Taking the lowest-order terms in system (3.21), (3.22), (3.23) yields the following set of equations:

$$\left.\begin{aligned} &\partial_t\mathscr{U}^{(0)} + \mathrm{i}\mathscr{U}^{(0)} + 2\partial_{\xi^*}p^{(0)} = 0,\\ &\partial_z p^{(0)} + \rho^{(0)} = 0,\\ &\partial_t\rho^{(0)} = 0,\\ &\partial_\xi\mathscr{U}^{(0)} + \partial_{\xi^*}\mathscr{U}^{(0)*} + \partial_z w^{(0)} = 0, \end{aligned}\right\} \tag{3.30}$$

with initial and boundary conditions

$$\left.\begin{aligned}&\left(\mathscr{U}_I^{(0)},\rho_I^{(0)}\right)=(\mathscr{U}_I,\rho_I),\\&w^{(0)}\big|_{z=-1,0}=0.\end{aligned}\right\}\tag{3.31}$$

The third equation in (3.30) means that the density field is slow: $\rho^{(0)}=\bar{\rho}_0(\boldsymbol{r};t_1,\ldots)$. The pressure field is split into the slow and the fast parts:

$$p=\bar{p}^{(0)}(\boldsymbol{r};t_1,\ldots)+\tilde{p}^{(0)}(\boldsymbol{r};t,\ldots).\tag{3.32}$$

By introducing the slow–fast decomposition of the velocity field we obtain the following equations for the slow and the fast components, respectively:

$$\bar{\mathscr{U}}^{(0)}=2\mathrm{i}\partial_{\xi^*}\bar{p}^{(0)},\quad \partial_z\bar{p}^{(0)}+\bar{\rho}^{(0)}=0,\quad \partial_\xi\bar{\mathscr{U}}^{(0)}+\partial_{\xi^*}\bar{\mathscr{U}}^{*(0)}+\partial_z\bar{w}^{(0)}=0,\tag{3.33}$$

$$\partial_t\tilde{\mathscr{U}}_0+\mathrm{i}\tilde{\mathscr{U}}_0=-2\partial_{\xi^*}\tilde{p}^{(0)},\quad \partial_z\tilde{p}^{(0)}=0,\quad \partial_\xi\tilde{\mathscr{U}}_0+\partial_{\xi^*}\tilde{\mathscr{U}}_0^*+\partial_z\tilde{w}^{(0)}=0.\tag{3.34}$$

The corresponding boundary conditions are

$$\bar{w}^{(0)}\big|_{z=-1,0}=\tilde{w}^{(0)}\big|_{z=-1,0}=0.\tag{3.35}$$

From (3.33), (3.35) it follows that $\bar{w}^{(0)}=0$. We represent the fast component of the horizontal velocity as a sum of baroclinic and barotropic parts:

$$\tilde{\mathscr{U}}^{(0)}=\tilde{\mathscr{U}}_{bc}^{(0)}(\boldsymbol{r},t,t_1,\ldots)+\tilde{\mathscr{U}}_{bt}^{(0)}(x,y,t,t_1,\ldots),\quad \int_{-1}^{0}\mathrm{d}z\tilde{\mathscr{U}}_{bc}^{(0)}=0.\tag{3.36}$$

Using (3.32), (3.34), and (3.35) we obtain for the barotropic component

$$\partial_t\tilde{\mathscr{U}}_{bt}^{(0)}+\mathrm{i}\tilde{\mathscr{U}}_{bt}^{(0)}=-2\partial_{\xi^*}\tilde{p}^{(0)},\quad \partial_\xi\tilde{\mathscr{U}}_{bt}^{(0)}+\partial_{\xi^*}\tilde{\mathscr{U}}_{bt}^{(0)*}=0,\tag{3.37}$$

whence it readily follows that $\tilde{\mathscr{U}}_{bt}^{(0)}=\tilde{p}^{(0)}=0$. We thus obtain

$$\tilde{\mathscr{U}}^{(0)}=\tilde{\mathscr{U}}_{bc}^{(0)}=\mathscr{A}^{(0)}(\boldsymbol{r};t_1,\ldots)\,\mathrm{e}^{-\mathrm{i}t},\quad \int_{-1}^{0}\mathrm{d}z\mathscr{A}^{(0)}=0\tag{3.38}$$

and

$$w^{(0)}=\tilde{w}^{(0)}=\mathscr{W}^{(0)}(\boldsymbol{r};t_1,\ldots)\,\mathrm{e}^{-\mathrm{i}t}+\text{c.c.},\tag{3.39}$$

where $\mathscr{W}^{(0)}=-\partial_\xi\mathscr{M}^{(0)}$. The function $\mathscr{M}^{(0)}$ is the primitive with respect to z of the modulated amplitude of the inertial oscillations:

$$\partial_z\mathscr{M}^{(0)}=\mathscr{A}^{(0)}.\tag{3.40}$$

Thus, at the lowest order the flow is split into slow geostrophic (vortical) motion with no vertical velocity and a fast inertial oscillations field. The former is completely determined by the quasi-steady pressure field and the latter has no signature in the pressure field. Regarding initial conditions, the slow density and pressure are in hydrostatic balance, $\partial_z\bar{p}_I^{(0)}=-\bar{\rho}_I$, and by decomposing the initial pressure field into the baroclinic and the barotropic parts:

$$\bar{p}_I^{(0)}=\bar{p}_{bc_I}^{(0)}(\boldsymbol{r})+\bar{p}_{bt_I}^{(0)}(x,y),\tag{3.41}$$

where $\int_{-1}^{0}\mathrm{d}z\,\bar{p}_{bc_I}^{(0)}=0$, we obtain

$$\bar{p}_{bc_I}^{(0)}=-\int_{-1}^{z}\mathrm{d}z\rho_I+\int_{-1}^{0}\mathrm{d}z'\int_{-1}^{z'}\mathrm{d}z\rho_I.\tag{3.42}$$

In order to find $\bar{p}^{(0)}_{bt_I}$ and $\bar{\mathscr{U}}^{(0)}_I$ we represent $\bar{\mathscr{U}}^{(0)}$ in the same form:

$$\bar{\mathscr{U}}^{(0)} = \bar{\mathscr{U}}^{(0)}_{bc}(\boldsymbol{r}, t_1, \ldots) + \bar{\mathscr{U}}^{(0)}_{bt}(x, y, t, t_1, \ldots) \tag{3.43}$$

with $\int_{-1}^{0} \mathrm{d}z\, \bar{\mathscr{U}}^{(0)}_{bc} = 0$. We have

$$\bar{\mathscr{U}}^{(0)}_I = \bar{\mathscr{U}}^{(0)}_{bc_I}(\boldsymbol{r}) + \bar{\mathscr{U}}^{(0)}_{bt_I}(x, y), \tag{3.44}$$

where $\bar{\mathscr{U}}^{(0)}_{bc_I} = 2\mathrm{i}\partial_{\xi^*}\bar{p}^{(0)}_{bc_I}$. Since $\mathscr{U}_I = \bar{\mathscr{U}}^{(0)}_I + \tilde{\mathscr{U}}^{(0)}_I = \bar{\mathscr{U}}^{(0)}_{bc_I} + \bar{\mathscr{U}}^{(0)}_{bt_I} + \tilde{\mathscr{U}}^{(0)}_I$ and $\int_{-1}^{0} \mathrm{d}z\, \tilde{\mathscr{U}}^{(0)}_I = 0$ the barotropic component at the initial moment is

$$\bar{\mathscr{U}}^{(0)}_{bt_I} = \mathscr{U}_{bt_I} = \int_{-1}^{0} \mathrm{d}z\, \mathscr{U}_I \tag{3.45}$$

and, therefore,

$$\tilde{\mathscr{U}}^{(0)}_I = \mathscr{A}^{(0)}_I = \mathscr{U}_I - \int_{-1}^{0} \mathrm{d}z\, \mathscr{U}_I - \bar{\mathscr{U}}^{(0)}_{bc_I}. \tag{3.46}$$

Finally, the field $\bar{p}^{(0)}_{bt_I}$ is determined from the equation

$$2i\,\partial_{\xi^*}\bar{p}^{(0)}_{bt_I} = \bar{\mathscr{U}}^{(0)}_{bt_I}. \tag{3.47}$$

3.3.4. *The second-order evolution equations*

At first order in ϵ the system (3.21) yields

$$\left.\begin{aligned}
&\partial_t \mathscr{U}^{(1)} + \mathrm{i}\mathscr{U}^{(1)} = -\big[\partial_{t_1}\mathscr{U}^{(0)} + 2\partial_{\xi^*}p^{(1)} + \big(\mathscr{U}^{(0)}\partial_\xi + \mathscr{U}^{(0)*}\partial_{\xi^*}\big)\mathscr{U}^{(0)} + w^{(0)}\partial_z\mathscr{U}^{(0)}\big],\\
&\partial_z p^{(1)} + \rho^{(1)} = 0,\\
&\partial_t \rho^{(1)} = -\big[\partial_{t_1}\rho^{(0)} + \big(\mathscr{U}^{(0)}\partial_\xi + \mathscr{U}^{(0)*}\partial_{\xi^*}\big)\rho^{(0)} + w^{(0)}\partial_z\rho^{(0)}\big],\\
&\partial_\xi \mathscr{U}^{(1)} + \partial_{\xi^*}\mathscr{U}^{(1)*} + \partial_z w^{(1)} = 0.
\end{aligned}\right\} \tag{3.48}$$

The evolution equation for the slow density field at this order

$$\partial_{t_1}\rho^{(0)} + \bar{\mathscr{U}}^{(0)}\partial_\xi\rho^{(0)} + \bar{\mathscr{U}}^{*(0)}\partial_{\xi^*}\rho^{(0)} = \partial_{t_1}\rho^{(0)} + J\big(\bar{p}^{(0)}, \rho^{(0)}\big) = 0 \tag{3.49}$$

is readily derived from the mass conservation equation in (3.48) by averaging over the fast time. Thus, the zeroth-order density field is horizontally advected by the geostrophic velocity. The fast part of the mass conservation equation in (3.48) gives

$$\partial_t \rho^{(1)} = -\big[\big(\tilde{\mathscr{U}}^{(0)}\partial_\xi + \tilde{\mathscr{U}}^{(0)*}\partial_{\xi^*}\big)\rho^{(0)} + \tilde{w}^{(0)}\partial_z\rho^{(0)}\big]. \tag{3.50}$$

Hence, the full density field at this order (cf. (3.39), (3.50)) is given by

$$\rho_1 = \overline{\rho}^{(1)}(\boldsymbol{r}; t_1, \ldots) + [\mathscr{R}_1(\boldsymbol{r}; t_1, \ldots)\,\mathrm{e}^{-\mathrm{i}t} + \text{c.c.}\ ], \tag{3.51}$$

whence

$$p^{(1)} = \overline{p}^{(1)}(\boldsymbol{r}; t_1, \ldots) + \big[\mathscr{P}^{(1)}(\boldsymbol{r}; t_1, \ldots)\,\mathrm{e}^{-\mathrm{i}t} + \text{c.c.}\ \big], \tag{3.52}$$

where

$$\partial_z\mathscr{P}^{(1)} = -\mathscr{R}^{(1)} = \mathrm{i}\big[\mathscr{A}^{(0)}\partial_\xi\rho^{(0)} + \mathscr{W}^{(0)}\partial_z\rho^{(0)}\big] = \mathrm{i}\frac{\partial\big(\mathscr{M}^{(0)}, \rho^{(0)}\big)}{\partial\,(z, \xi)} \tag{3.53}$$

and we have used another standard notation for Jacobians allowing us to display explicitly the independent variables.

As usual in the frontal regime, the right-hand side of the horizontal velocity equation in (3.48) contains resonances. Their removal gives an evolution equation for the inertial oscillations envelope:

$$\partial_{t_1}\mathscr{A}^{(0)} - 2\mathrm{i}\frac{\partial\left(\bar{p}^{(0)}, \mathscr{A}^{(0)}\right)}{\partial\left(\xi, \xi^*\right)} + 2\mathrm{i}\mathscr{A}^{(0)}\partial^2_{\xi\xi^*}\bar{p}^{(0)} + 2\mathrm{i}\mathscr{W}^{(0)}\partial_z\partial_{\xi^*}\bar{p}^{(0)} + 2\partial_{\xi^*}\mathscr{P}^{(1)} = 0. \tag{3.54}$$

Differentiating (3.54) in z and using (3.53) and the definition of $\mathscr{M}^{(0)}$ we arrive at the following modulation equation for $\mathscr{M}^{(0)}$:

$$\partial_{t_1}\partial_{zz}\mathscr{M}^{(0)} - 2\mathrm{i}\frac{\partial\left(\bar{p}^{(0)}, \partial^2_{zz}\mathscr{M}^{(0)}\right)}{\partial\left(\xi, \xi^*\right)} + 2\mathrm{i}\partial^2_{\xi\xi^*}\bar{p}^{(0)}\partial^2_{zz}\mathscr{M}^{(0)} \\ - 2\mathrm{i}\left(\partial_z\rho^{(0)}\right)\partial^2_{\xi\xi^*}\mathscr{M}^{(0)} - 4\mathrm{i}\left(\partial^2_{\xi z}\bar{p}^{(0)}\right)\partial^2_{\xi^* z}\mathscr{M}^{(0)} = 0, \tag{3.55}$$

which can be rewritten in Cartesian coordinates in the form (3.29).

This equation (with additional terms arising on the β-plane) was initially derived by Young & Ben Jelloul (1997) and then studied by Balmforth & Young (1999). Young & Ben Jelloul omitted the last term, arguing that the shear of the background geostrophic flow in their setting was weak. However, they also pointed out that this term was needed to ensure energy conservation for the inertial oscillations field.

Once resonances are removed, the first correction $\mathscr{U}^{(1)}$ to the (complex) horizontal velocity field is readily calculated from the first equation in (3.48). Then the first correction to the vertical velocity can be easily determined from the continuity equation in (3.48). Its slow part is given by

$$\partial_z\overline{w}^{(1)} = -2\partial_{t_1}\partial^2_{\xi\xi^*}\bar{p}^{(0)} - 2\mathrm{i}\partial^2_{\xi\xi^*}\bar{p}^{(1)} + 4\mathrm{i}\partial_\xi\left(\partial_{\xi^*}\bar{p}^{(0)}\partial^2_{\xi\xi^*}\bar{p}^{(0)} - \partial_\xi\bar{p}^{(0)}\partial^2_{\xi^*\xi^*}\bar{p}^{(0)}\right) \\ - \mathrm{i}\partial_\xi\left(\mathscr{A}^{(0)^*}\partial_{\xi^*}\mathscr{A}^{(0)} - \partial_{\xi^*}\mathscr{M}^{(0)*}\partial_z\mathscr{A}^{(0)}\right) + \text{c.c.} \tag{3.56}$$

By integrating this equation over z from -1 to 0 and using the definition of $\mathscr{M}^{(0)}$ and the vertical boundary conditions we obtain (3.27). Finally, we rewrite (3.49) in terms of $\bar{p}^{(0)}$ and obtain (3.28).

We thus see that at this order the evolution of the system is described by two autonomous equations for the slow pressure coupled with a modulation equation for inertial oscillations field. The modulation equation (3.55) describes the redistribution of energy among the inertial oscillations 'catalyzed', in the language of Lelong & Riley (1991), by the vortical component. It is worth repeating that, just as in the two-layer FGH regime (and unlike the one-layer FG and two-layer FGI), the slow component and the fast oscillations envelope evolve in the same time t_1.

3.3.5. *The third-order approximation*

As before, we will not dwell on the tedious, but rather straightforward, calculations of the third-order corrections to the velocity and density (or pressure) fields, limiting ourselves to analysis of the PV equation. We should mention, however, that cubic corrections to the modulation equation (3.29) identically vanish, as in one- and two-layer FG regimes.

The potential vorticity $\varPi$ in the FG context is expressed as

$$\varPi = \partial_z\rho + \epsilon[\zeta\,\partial_z\rho - \partial_z v\partial_x\rho + \partial_z u\partial_y\rho]. \tag{3.57}$$

Let us state the result of the calculations at the previous orders of the perturbation theory. At the lowest order we find that $\varPi^{(0)} = \partial_z\rho^{(0)}$ does not depend on the fast time. At the first order in ϵ we obtain from (3.11)

$$\partial_t\varPi^{(1)} + \partial_{t_1}\varPi^{(0)} + \boldsymbol{v}^{(0)}\cdot\nabla\varPi^{(0)} = 0, \tag{3.58}$$

where

$$\Pi^{(1)} = \partial_z \rho^{(1)} + \zeta^{(0)} \partial_z \rho^{(0)} - \partial_z v^{(0)} \partial_x \rho^{(0)} + \partial_z u^{(0)} \partial_y \rho^{(0)}. \tag{3.59}$$

Note that the inertial oscillations give rise to PV at the first order as

$$\Pi^{(1)} = \partial_z \rho^{(0)} - \mathrm{i} \frac{\partial\left(\rho^{(0)}, \mathscr{U}^{(0)}\right)}{\partial\left(z, \xi\right)} + \text{c.c.} \tag{3.60}$$

Finally, at the third order we have the equation

$$\partial_t \Pi^{(2)} + \partial_{t_1} \Pi^{(1)} + \partial_{t_2} \Pi^{(0)} + \boldsymbol{v}^{(0)} \cdot \nabla \Pi^{(1)} + \boldsymbol{v}^{(1)} \cdot \nabla \Pi^{(0)} = 0. \tag{3.61}$$

Averaging this equation over the fast time t gives

$$\partial_{t_1} \bar{\Pi}^{(1)} + \partial_{t_2} \bar{\Pi}^{(0)} + \bar{\boldsymbol{v}}^{(0)} \cdot \nabla \bar{\Pi}^{(1)} + \bar{\boldsymbol{v}}^{(1)} \cdot \nabla \bar{\Pi}^{(0)} + \left\langle \tilde{\boldsymbol{v}}^{(0)} \cdot \nabla \tilde{\Pi}^{(1)} \right\rangle = 0. \tag{3.62}$$

Tedious but straightforward calculations using the Jacobi identity for the Jacobians show that the fast-component contributions containing the envelope of the inertial oscillations $\mathscr{A}^{(0)}$ in the third and the fifth terms in (3.62) are mutually cancelled, while they are absent in the fourth term. Thus, there is no inertial-oscillations drag in the PV evolution equation. By introducing the 'full' fields

$$\bar{p} = \bar{p}^{(0)} + \epsilon \bar{p}^{(1)}, \tag{3.63}$$

$$\bar{\Pi} = -(1 + \epsilon \nabla^2 \bar{p}) \partial_{zz}^2 \bar{p} + \epsilon |\nabla \partial_z \bar{p}|^2 \tag{3.64}$$

the PV evolution equation may be rewritten in an explicitly conservative form:

$$\partial_\tau \overline{\Pi} + \bar{\boldsymbol{v}} \cdot \nabla \overline{\Pi} = O(\epsilon^2), \tag{3.65}$$

where the slow divergenceless velocity field $\bar{\boldsymbol{v}}$ is given by the complete geostrophic velocity plus slow ageostrophic corrections. These corrections are provided by the slow pressure terms in $\bar{\mathscr{U}}^{(1)}$, $\bar{w}^{(1)}$, the terms containing self-interaction of the fast component being cancelled as explained above.

4. Discussion

The problem of nonlinear geostrophic adjustment or, more generally, the problem of interaction between the fast and the slow components of motion is of fundamental importance for understanding rotating stratified turbulence and has been under active study for many years (see e.g. Bartello 1995 and references therein). Practically, its solution is crucial for weather and climate prediction (see Part 1 for general discussion). This problem is attracting much attention from the mathematical community and many important results have recently been obtained (Babin, Mahalov & Nicolaenko 1996, 1999, 2000; Babin *et al.* 1997; Embid & Majda 1996, 1998). All these works deal with a spatially periodic motion in a periodic box containing a 'soup' of wave-modes and 'bones' of vortex-modes evolving together. This setting is clearly well-suited for comparisons with numerical simulations of turbulence and allows, in particular, application of the ideas of statistical equilibrium to find the evolution trends (cf. Bartello 1995).

In the present paper we take a different approach, however, which is more appropriate when studying geostrophic adjustment. Namely, we consider the problem of the evolution of a localized initial disturbance in the open domain, thus letting the fast waves be radiated to infinity. In terms of comparison with numerical simulations, therefore, our results should correspond to the sponge boundary conditions in the horizontal directions. This approach allows us to avoid wave–wave resonances which were thoroughly studied in the above-mentioned papers. The only resonances which do play a role in our analysis are those due to the 'catalytic' interactions (Lelong & Riley 1991; Bartello 1995) of quasi-stationary inertial oscillations with the vortical flow in the frontal geostrophic regime. Inertial oscillations are non-propagative and, therefore, they remain for a long time at the initial perturbation location. However, they do not make any contribution to the equations governing the slow motion, which we confirm by direct calculation. Another idealization is that we use the hydrostatic approximation compatible with a thin slab geometry, leaving the study of non-hydrostatic Euler–Boussinesq equations for a future work. This approximation simplifies the analysis because of the absence of vertically propagating inertia waves which can produce resonances due to the smallness of their horizontal group velocities (cf. Babin, Mahalov & Nicolaenko 1998).

The main result of the present work is that, like in the barotropic case considered in Part 1, the motion of rapidly rotating (small Rossby number) stratified fluid is uniquely split into the slow part close to the geostrophic balance and the fast part consisting of rapidly propagating IGW or non-propagating inertial oscillations. The key point is that the fast component does not affect the slow one up to the third order in the Rossby number. At the same time, the adjustment scenario strongly depends on the stratification and the order of magnitude of typical deviations of the isopycnal surfaces from their equilibrium positions. We considered above the geostrophic regimes for two types of stratification of the fluid contained in a slab: a density jump (two-layer model) and a smooth density profile (continuously stratified model). If the deviations of the isopycnal surfaces (of the interface in the two-layer case) are small, the adjustment follows the quasi-geostrophic scenario. The slow component is governed by the standard PV equations on times of the order of $(fRo)^{-1}$. The method we use allows to proceed further in the expansion in Ro and to derive an improved quasi-geostrophic PV equation describing the slow QG component on much longer times of the order of $(fRo^2)^{-1}$. This equation is also written in PV-conservation form and is reduced to the standard quasi-geostrophic PV equation for times $O(f^{-1}Ro^{-1})$. The fast component consists of the internal IGW rapidly propagating outward from the localized initial perturbation. It decays in time at a fixed spatial location. Nonlinear interactions of the fast waves with each other and with the slow component do not produce any drag in the slow-component equations (at least up to times $O(f^{-1}Ro^{-2})$).

In the case of strong deviations of the isopycnal surfaces all the fields, again, are split into slow and fast components, each of them evolving from the well-defined initial conditions. But unlike the QG case the fast component here consists of non-propagating inertial oscillations with slowly modulated amplitude. The modulation is described by a Schrödinger-type equation with coefficients depending on the slow variables, i.e. the oscillations are strongly coupled to the slow component, unlike the QG case. Another distinction from the QG case is that nonlinear self-interaction of the fast component does give rise to the slow velocity field. However, and this is a non-trivial fact, this interaction *does not result in any drag terms* in the equations for the slow component.

Correspondingly, the slow component is governed by the frontal geostrophic dynamics equations derived in the papers cited in the Introduction. In the two-layer model the frontal regime depends strongly on the density stratification, i.e. on the ratio of the layer depths. We distinguish between two basic cases: homogeneous FGH (layer depths of the same order) and inhomogeneous FGI (thin upper layer). In the FGH sub-regime the slow component and the modulation amplitude of the inertial oscillations evolve in the same slow time $t_1 = Ro\,t$ while in the FGI regime the slow component is even slower and evolves in $t_2 = Ro^2 t$, while the modulation amplitude still evolves in t_1. Thus, for large-scale initial disturbances with strong density perturbations the geostrophic adjustment may be incomplete or delayed due to the presence of the fast near-inertial oscillations co-evolving with the slow frontal-geostrophic vortical component of the flow.

In the continuously stratified fluid the frontal regime is similar to the two-layer FGH sub-regime as the slow component and the envelope of the inertial oscillations both evolve in t_1. It should be noted, however, that while performing our calculations we tacitly assume that the initial density has no discontinuities nor sharp gradients in the vertical. If this assumption does not hold, the asymptotic procedure of § 3.3 should be modified. We believe that regimes analogous to the FGI one are also possible for certain continuous stratifications, for example the thermocline-type stratifications with a thin sharp thermocline.

The perturbation method we use imposes obvious self-consistency restrictions and has a validity domain limited in time. The motion should preserve its single-scale character for both fast and slow components (including the vertical scale in continuously stratified model) and there should not be explosive finite-time instabilities in slow dynamics destroying the slow-time scaling.

The limits of the single space-scale approach become evident when passing to the β-plane equations. The β-effect may be introduced in all models above along the lines of Part 1 with the same conclusions. First, the fast–slow splitting persists on the β-plane, too, and the slow-component equations are the same as those derived by filtering of the fast component. Thus, the slow dynamics of the stratified fluid on the β-plane is self-consistent. Second, the single spatial-scale asymptotic approach results in incurable secular growth of the higher-order fast corrections. A self-consistent asymptotic procedure, yet to be developed, should be based on at least two spatial scales in order to take into account distortions of the fast-wave rays due to the spatial inhomogeneity induced by the β-effect.

Another problem is baroclinic instability which should be endemic in the frontal regimes, whose characteristic scales are much larger than the deformation radius. It is evident that this instability can destroy the presumed FG scaling if it gives rise to the growth of perturbations with scale of the order of the deformation radius. Benilov (1993) showed that in the continuously stratified frontal regime the instability occurs for perturbations with small enough horizontal scale. Moreover, the instability growth rate tends to infinity as the perturbation scale goes to zero. Stability of the two-layer frontal geostrophic regimes was considered by Benilov & Cushman-Roisin (1994). Swaters (1993) explored a similar problem for the two-layer ocean with sloping bottom. Karsten & Swaters (2000*a*, *b*) investigated in detail the stability of various two-layer frontal sub-regimes on the β-plane. The results of these papers indicate that higher-order corrections to the FG slow equations (like those calculated above for the QG-regime) are necessary to control the instability. In any case our analysis, as in the RSW case in Part 1, shows that the evolution properties of slow structures, in general, and their stability, in particular, may be safely studied within the slow dynamics equations as they are not influenced at all by the fast component of the flow. Even if the flow in the frontal regime develops a baroclinic instability, unless this is explosive one can still follow the early stages (note that 'early' is in the slow-time sense and that the baroclinic instability is known to be slow to develop) of the joint evolution of inertial oscillations and vortex flow with the help of the fast and slow FG equations.

We should also emphasize that the 'improved' QG equations obtained in Part 1 and in the present paper deserve further study as they contain higher derivatives than the standard QG equations which should modify the short-scale behaviour of solutions. It should be noted that the improved QGPV equations in RSW, 2RSW and HSPE may be simplified by changing from the geostrophic pressure(s) to the geostrophic Bernoulli function(s) $B = p - \frac{1}{2}\epsilon(\nabla p)^2$. (The usefulness of the Bernoulli function in the context of balanced motions was advocated long ago by Sutyrin 1994). In this way the gradients disappear from the advecting velocity and the third derivatives disappear from the advected PV, which is important for computational purposes. It should be also mentioned that Vallis (1996) obtained a system of balanced equations from 'a step beyond quasi-geostrophy' approximation by direct filtering of the fast component and by considering corrections to the QG PV-inversion in the primitive equations in the atmospheric context. We believe that this system may be further reduced to a single improved QGPV equation.

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